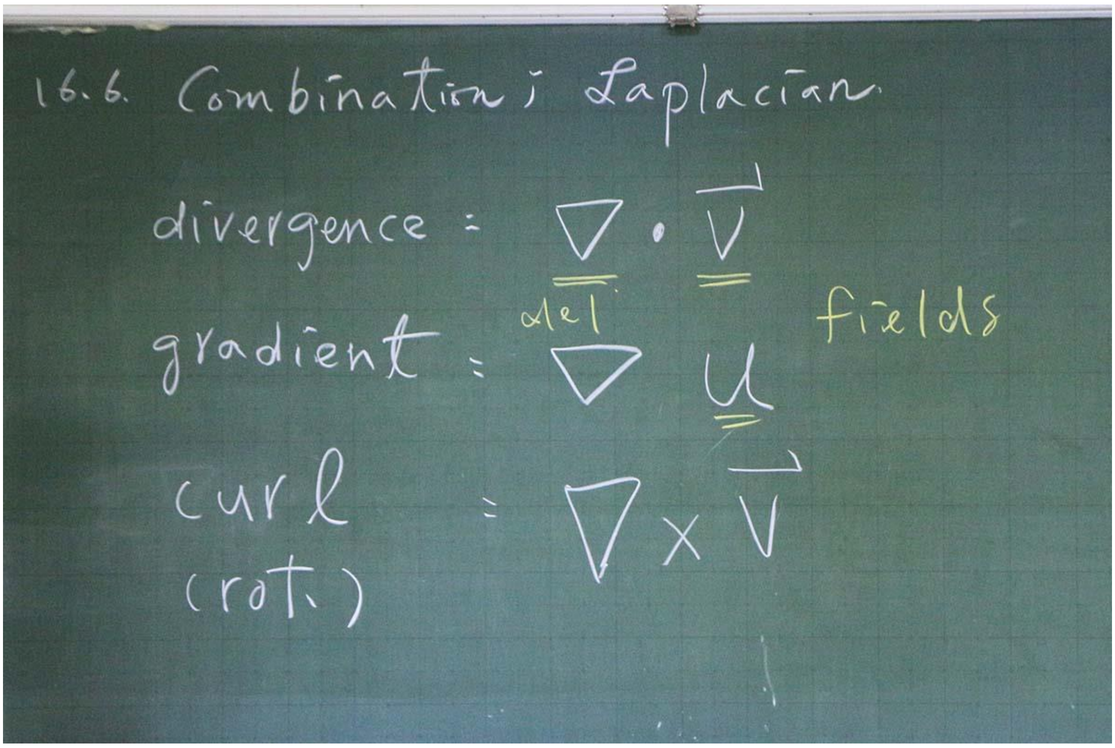

gradient = ∇ u
curl = ∇
(rot.)


16.6. Combination; Laplacian
divergence = ∇ · V
del
gradient = ∇ u
fields
curl = ∇ × V
(rot.)

I. An operator on a combination of fields. (x8)

<e.g.> $\nabla \cdot (\underbrace{u\vec{V}}_{\text{combination}}) = \nabla u \cdot \vec{V} + u \nabla \cdot \vec{V}$

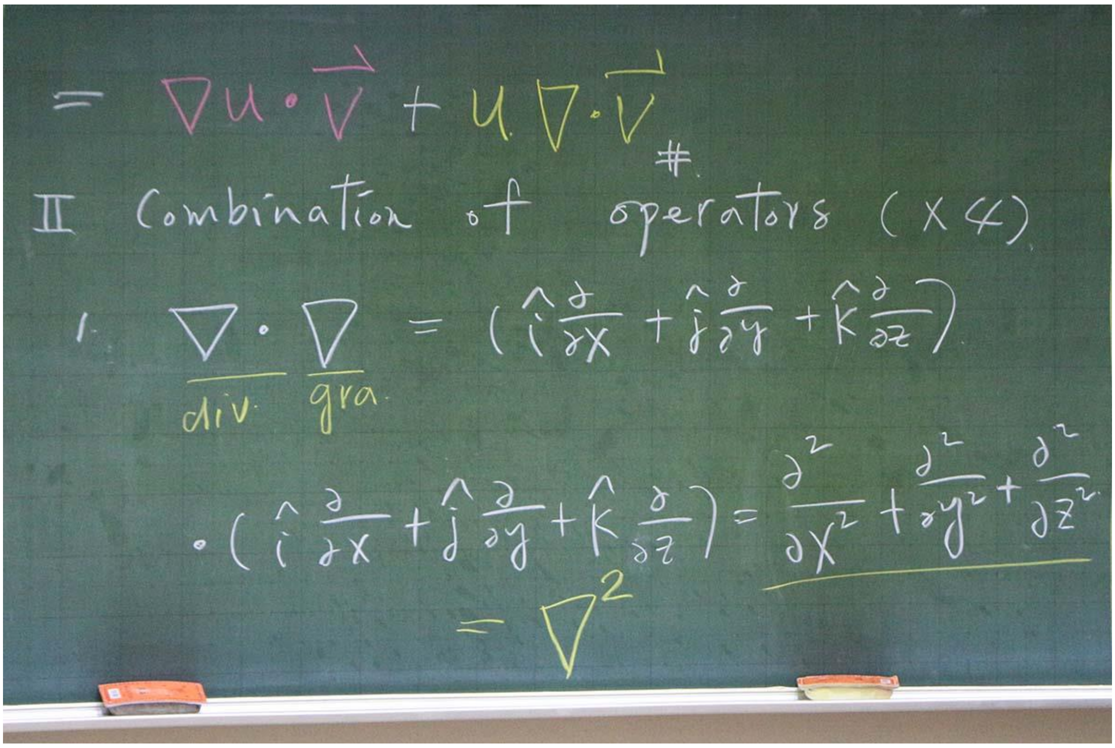
pf $\underbrace{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial x}\right)}_{3} \cdot \underbrace{\left(uV_x\hat{i} + uV_y\hat{j} + uV_z\hat{k}\right)}_{3}$

$$= \frac{\partial}{\partial x}(uV_x) + \frac{\partial}{\partial y}(uV_y) + \frac{\partial}{\partial z}(uV_z)$$

$$= \left(\frac{\partial u}{\partial x} V_x + u\frac{\partial V_x}{\partial x}\right) + \left(\frac{\partial u}{\partial y} V_y + u\frac{\partial V_y}{\partial y}\right)$$

$$+ \left(\frac{\partial u}{\partial z} V_z + u\frac{\partial V_z}{\partial z}\right) = \left(u_x V_x + u_y V_y + u_z V_z\right)$$

$$+ \left(u\frac{\partial V_x}{\partial x} + u\frac{\partial V_y}{\partial y} + u\frac{\partial V_z}{\partial z}\right)$$


$= \nabla u \cdot \vec{V} + u \nabla \cdot \vec{V}$ #
II Combination of operators (×4)
1. $\nabla \cdot \nabla = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})$
div. gra.
$\cdot (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$= \nabla^2$


$+ \frac{\partial}{\partial y}(uV_y) + \frac{\partial}{\partial z}(uV_z)$
$= (U_xV_x + U_yV_y + U_zV_z)$
$= \nabla u \cdot \vec{V} + u \nabla \cdot \vec{V}$ #
II Combination of operators (×4)

① $\nabla^2 \equiv$ Laplacian $= \triangle$

2. $$\underbrace{\nabla \cdot}_{\text{div.}} \underbrace{\nabla \times}_{\text{curl}} \vec{V} = 0$$

PS.

② ∇^2 can be applied on both scalar & vector fields!!

e.g., (i) $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(ii) $$\nabla^2 \vec{V} = \underline{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}\left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k}\right)$$

$$= \hat{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)$$

$$+ \hat{j}\left(\quad \sim \quad\right)$$

$$+ \hat{k}\left(\quad \sim \quad\right)$$

3. center gravity

$$\underbrace{\nabla \times}_{\text{curl}} \underbrace{\nabla}_{\text{grd.}} U = 0$$

vector

4. $$\underbrace{\nabla \times}_{\text{curl}} \underbrace{\nabla \times}_{\text{curl}} \vec{V} = \nabla \underbrace{\left(\nabla \cdot \vec{V}\right)}_{\text{div}} - \underbrace{\nabla^2 \vec{V}}_{\text{vector}}$$

vector

Maxwell's Eqn.

$$\nabla \cdot \vec{D} = \rho_v \quad ; \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \nabla \times \vec{H} = \vec{J}_c + \underline{\vec{J}_D}$$

$$= \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$$

$$\underline{\nabla \times \nabla \times \vec{E}} = \underline{\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}} \text{ ——— (1)}$$

LHS : $\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$ $\quad \mu \vec{H}$

$$= -\nabla \times \left(\frac{\partial\, \mu \vec{H}}{\partial t} \right) \quad \frac{\text{isotropic}}{\text{homogeneous}} \quad -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

$$= -\mu \frac{\partial}{\partial t}\left(\cancel{\vec{J}_c}^{\,0} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

source free

$$\text{———— } 2(a)$$

RHS :

$$\nabla\left(\cancel{\frac{\rho_v}{\varepsilon}}^{\,0}\right) - \nabla^2 \vec{E}$$

$$= -\nabla^2 \vec{E} \qquad \text{———— } 2(b)$$

2(a) = 2(b),

$$\frac{1}{\mu\varepsilon}\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \quad \#$$

so-called "Wave Eqn."

$$\frac{1}{\mu\varepsilon} = \frac{1}{\mu_0\mu_r}\cdot\frac{1}{\varepsilon_0\varepsilon_r} = \frac{1}{\mu_0\varepsilon_0}\,\frac{1}{\mu_r\varepsilon_r}$$

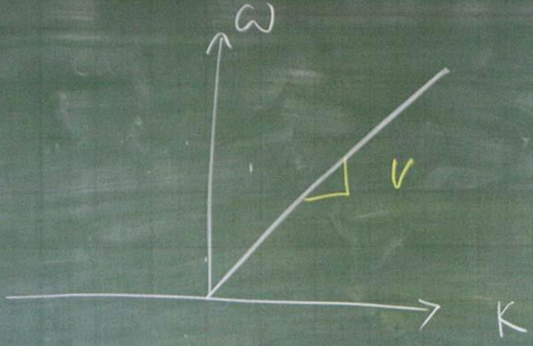
$$= c^2 \cdot \frac{1}{n^2}$$

$$= \frac{c^2}{n^2} = v^2$$

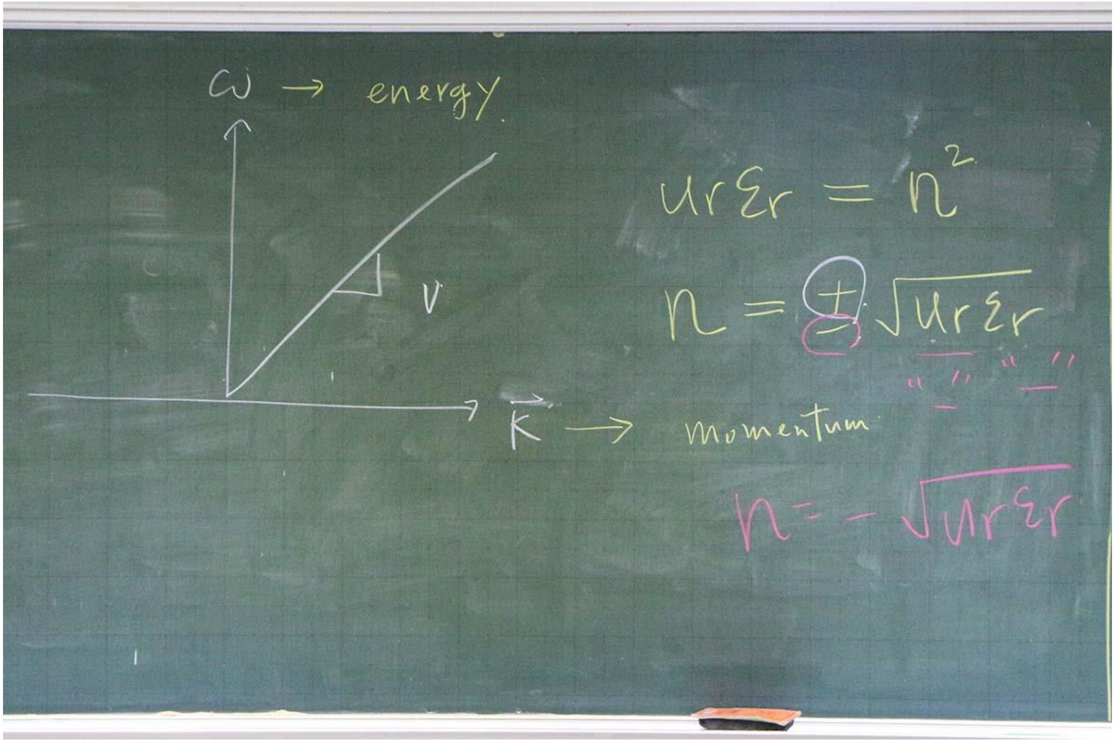
$$\vec{E} \propto \exp i(\vec{k}\cdot\vec{r} - \omega t)$$

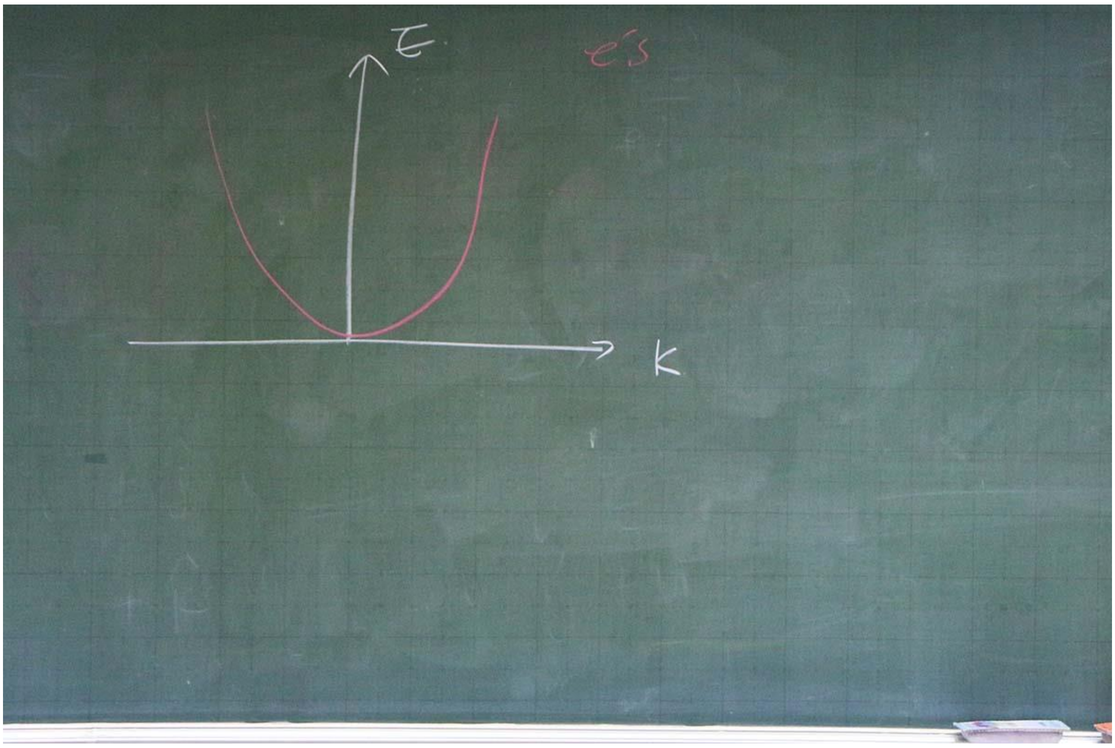
$$\frac{1}{\mu\varepsilon}\cdot(ik)^2\,\cancel{\vec{E}} = (-i\omega)^2\,\cancel{\vec{E}}$$

$$\mu\varepsilon\,\omega^2 = k^2 \rightarrow \omega = \sqrt{\frac{1}{\mu\varepsilon}}\,k = v k$$



dispersión !!


ω → energy
V
urεr = n²
n = ± √urεr
"+" "−"
k → momentum
n = −√urεr


E
e's
k

16.7. Non-Cartesian Coordinates

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

I. cylindrical coordinates

$$\begin{cases} (\hat{i}, \hat{j}, \hat{k}) \to (\hat{e}_r, \hat{e}_\theta, \hat{e}_z) \\ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \to \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right) \end{cases}$$

(i).

$$\hat{i} = \underline{\cos\theta\,\hat{e}_r} - \underline{\sin\theta\,\hat{e}_\theta} = c\,\hat{e}_r - s\,\hat{e}_\theta$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = s\,\hat{e}_r + c\,\hat{e}_\theta$$

$$\hat{k} = \hat{e}_z$$

(ii)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} \to 0$$

$$= r_x \frac{\partial}{\partial r} + \theta_x \frac{\partial}{\partial \theta} + 0 = c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \cdots$$

$$\frac{\partial}{\partial z} = \cdots$$

$$\therefore\ \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \boxed{\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}}$$

(ii) $\nabla u = \dfrac{\partial u}{\partial r}\hat{e}_r + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\hat{e}_\theta + \dfrac{\partial u}{\partial z}\hat{k}$

(ii) $\nabla \cdot \vec{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}\right)$

$\cdot \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z\right)$

Note

$$\hat{e}_r \underbrace{\frac{\partial}{\partial r}}_{1^{st}} \underset{=}{\cdot} V_r \hat{e}_r = \hat{e}_r \cdot \left[\frac{\partial}{\partial r}\left(V_r \hat{e}_r\right)\right]$$

$$\overset{2^{nd}}{=} \hat{e}_r \cdot \left[\frac{\partial V_r}{\partial r}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial r}\right]$$

16.6 Combi-nation

16.7 Non-Cartesian coordinates

16.8 Divergence theorem

6:40pm classes!!

$$= \frac{\partial V_r}{\partial r} + 0 = \frac{\partial V_r}{\partial r} \ \#.$$

right-handed orthogonal.

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & V_\theta & V_z \end{vmatrix} =$$

Spherical coordinates

$$\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

① 10.7 "change of bases"

1. $(\hat{i}, \hat{j}, \hat{k}) \longrightarrow (\hat{e}_\rho, \hat{e}_\phi, \hat{e}_\theta)$

② projection

2. $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \longrightarrow \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}\right)$

$$\hat{i} = (\hat{i}\cdot\hat{e}_\rho)\hat{e}_\rho + (\hat{i}\cdot\hat{e}_\phi)\hat{e}_\phi + (\hat{i}\cdot\hat{e}_\theta)\hat{e}_\theta$$

16.8 Divergence Theorem

$$\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\longrightarrow \int_V \nabla\cdot\vec{V}\, dV = \int_S \hat{n}\cdot\vec{V}\, dA$$

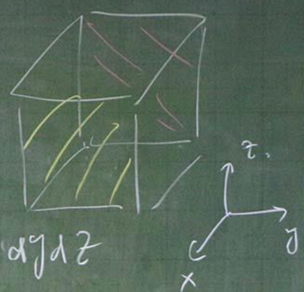
Pf

$$\int_V \nabla\cdot\vec{v}\, dV = \iint$$

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)dx\,dy\,dz$$

$$\iint_{x=x_2\text{ face}} V_x(x_2,y,z)\underbrace{dy\,dz}_{dA} - \iint_{x=x_1\text{ face}} V_x(x_1,y,z)\underbrace{dy\,dz}_{dA}$$

, where



16.6 Combi-nation

16.7. Non-Cartesian coordinates

16.8. Divergence theorem

6:40pm classes!!

$$= \iint_{x_2} \hat{i}\cdot\vec{v}\,dA + \iint_{x_1} (-\hat{i})\cdot\vec{v}\,dA$$

$$= \iint_{\substack{x=x_1\\x=x_2}} \hat{n}\cdot\vec{v}\,dA$$; by repeating this process, we derive,

$$\int_{v} \nabla\cdot\vec{v}\, d\bar{V} = \int_{S} \hat{n}\cdot\vec{v}\, dA \quad \#$$

<Ex.1>

Evaluate $\iint_S \vec{V}\cdot\hat{n}\, dA$, where S denotes the surface of $(x^2+y^2+z^2)=a^2$, and $\vec{V} = x\hat{i}+y\hat{j}+z\hat{k}$

Sol.

$$\iint_S \hat{n}\cdot\vec{V}\, dA = \iiint_{V} (\nabla\cdot\vec{V})\, d\bar{V}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\, \underbrace{\rho^2 |\sin\phi|}_{J(\rho,\phi,\theta)}\, d\rho\, d\phi\, d\theta$$

$$= 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3 \quad \#.$$

<Ex.2>

$$\int_S \hat{n}\cdot\vec{E}\, dA = \begin{cases} 0, & q = 0 \\ q_v/\varepsilon, & q \neq 0 \end{cases}$$